Vector Interpolation on Natural Element Method

Diego Pereira Botelho[1,2], Yves Marechal[1,2], and Brahim Ramdane[1,2]

[1]Univ. Grenoble Alpes, G2Elab, F-38000 Grenoble, France
[2]CNRS, G2Elab, F-38000 Grenoble, France

Electromagnetic simulation often requires vector edge-based shape functions. In a Finite Element Method (FEM) context, vector shape functions are designed for keeping the tangential continuities and normal discontinuities between elements, which corresponds to the physical nature of electromagnetic fields. However, in the *meshless methods* context, the construction of vector shape functions is not direct and very little work has been published on vector interpolation. This work is a first proposal of a NEM-based vector interpolation for polygonal cells. Fundamental results of the developed approach are demonstrated.

***Index Terms*—meshless methods, natural element method, polygonal elements, vector interpolation.**

I. Introduction

The proper field modeling on electromagnetics imposes some constraints that are not naturally met by nodal-based interpolation [1]. In the Finite Element Method (FEM) this issue is thoroughly settled by the use of edge-based interpolation [2]. The inherent properties of tangential continuity and normal discontinuity on interfaces found on the edge-based interpolation perfectly match the physical properties of electromagnetic fields.

When trying to develop vector interpolation in a meshless context, some of the characteristics found in FEM, like strict interpolation and gradient discontinuity on interfaces are missing. Besides that, meshless methods by definition also do not offer any kind of geometrical support for edge-based interpolation. The Natural Element Method (NEM) is a relatively recent numerical technique whose shape functions give accuracy typical of meshless methods while keeping the cited properties of FEM shape functions. Also NEM shape functions construction is based on a kind of mesh structure which can eventually provide geometrical basis for edges definition. For these reasons a vector interpolation based on NEM seems to be feasible. The goal of this work is to construct vector interpolation out of Natural Element Method (NEM) [3] shape functions.

This paper is organized as follows: the section II briefly presents the NEM and its main characteristics. Section III addresses the adopted approach to reach a vector interpolation. Section IV presents some fundamental results and section V concludes about the presented approach and sets the perspectives to the extended version of this paper.

II. Natural Element Method

The natural element method makes use of the *Voronoï diagram* for the construction of its shape functions. A Voronoï diagram is a subdivision of a domain into cells, where each cell is associated to a node in such way that any point inside a cell is closer to its respective node than to any other.

Fig. 1 depicts a Voronoï diagram for a set of nodes n_i in a 2D space. The *natural neighbors* of a given node are all the other nodes that share a facet of its Voronoï cell.

Data interpolation based on Voronoï diagrams can be reached through different techniques [4]. In this work the scheme proposed by Sibson [5] will be presented.

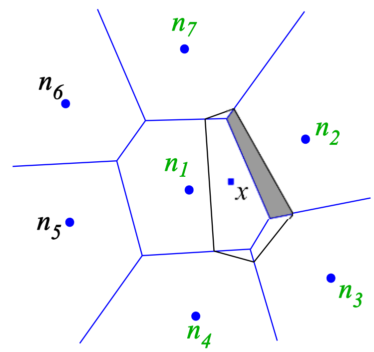

Fig. 1. Voronoï diagram in blue; Sibson interpolation: evaluation point $\boldsymbol{x}$, natural neighbors in green, additional Voronoï cell due to $\boldsymbol{x}$ in black and intersection area between Voronoï cells of $\boldsymbol{x}$ and n_2 in gray (the correspondent to $A_2(\boldsymbol{x})$).

Considering a set of nodes with known state variable values, the interpolation on an evaluation point $\boldsymbol{x}$ starts from the inclusion of this point into the original Voronoï diagram. Then a new Voronoï cell is created around $\boldsymbol{x}$ (Fig. 1). The natural neighbors of $\boldsymbol{x}$ (i.e. the closest nodes) are the nodes that share a common Voronoï facet with the cell of $\boldsymbol{x}$. These will be the nodes used for the interpolation at the evaluation point.

The Sibson shape functions are given by (1), where each $A_i(\boldsymbol{x})$ represents the intersection area between the Voronoï cell of $\boldsymbol{x}$ and the (original) Voronoï cell related to the natural neighbor n_i, as illustrated by the hashed region in Fig. 1.

$$\Phi_i(\boldsymbol{x}) = \frac{A_i(\boldsymbol{x})}{\sum_j A_j(\boldsymbol{x})} \tag{1}$$

The shape function (1) presents the following key properties:

$$\Phi_i(\mathbf{x}_j) = \delta_{ij} \,, \sum_{i=1}^{n} \Phi_i(\mathbf{x}) = 1 \text{ and } \sum_{i=1}^{n} \Phi_i(\mathbf{x})\mathbf{x}_i = \mathbf{x} \tag{2}$$

Furthermore, Sibson shape functions degenerate to a simple linear interpolation at the domain borders [3].

III. Vector Shape Functions on NEM

Since properties in (2) are settled, it seems natural to apply the same general process that generates vector shape functions out of the regular scalar shape functions on FEM [6]. However, when applied to polygons other than triangles and quadrangles, this approach requires more basis functions than the number of (external) edges of the element [8].

In order to overcome this inconvenient an alternative method has been proposed very recently in [9]. In that work a vector basis function is obtained through a discrete Helmholtz decomposition. This function, when applied to generalized barycentric coordinates, yields vector interpolation on polygonal elements that satisfies the desired properties with the minimal amount of degrees of freedom (one per edge) [9]. Moreover the method is extensible to three dimensions. The scheme was implemented in the present work in 2D with NEM interpolants, providing excellent results.

In our implementation a mesh of polygons was used as support for the interpolation. The Voronoï diagram was created for the nodes at the polygon's vertices. On the borders of the polygons the Voronoï cells were truncated in a C-NEM way [7].

Fig. 2 presents the vector fields associated to the highlighted edges for three different polygons.

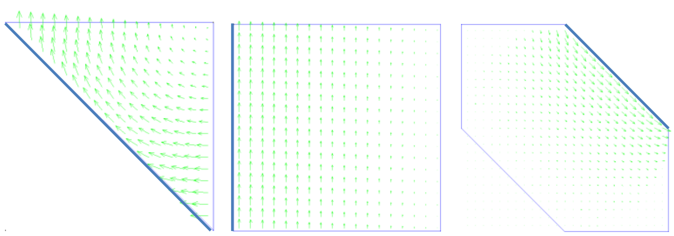

Fig. 2. Vector field associated to the highlighted edges for three different polygons: triangle, rectangle and hexagon.

For the tested polygons the same properties found in the classic FEM vector interpolation in terms of shape functions integration on borders are reached. These properties can be written as:

$$\int_{e_a} We_a(\mathbf{x})dl = 1 \text{ and } \int_{e_b} We_a(\mathbf{x})dl = 0 \text{ (for } b \neq a) \qquad (3)$$

IV. Fundamental Results

A validation of the proposed scheme was performed through the interpolation of vector fields. Two field configurations were tested over a regular mesh with only rectangles and an irregular one with different polygons configurations. The meshed domain in all cases is a unitary sized square centered at the point (0.0, 0.0).

The first case (Fig. 3) tests the ability of the method in reproducing a uniform field. The applied field is rotated 45 degrees in relation to the horizontal line. It can be verified that even with the coarse meshes tested the method was capable of reproducing in a good measure a uniform field. The mean relative errors calculated for the regular and the irregular meshes were 1.08E-9 % and 1.33E-6 %, respectively.

Fig. 4 shows the interpolation of a curl field generated by a punctual source placed at the point (-1.0, 0.0). The found errors were 6.27 % for the regular mesh and 6.68 % for the irregular one.

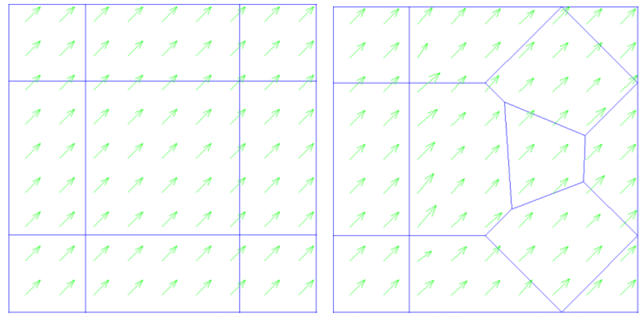

Fig. 3. Uniform magnetic field interpolated on two different meshes through vector NEM shape functions.

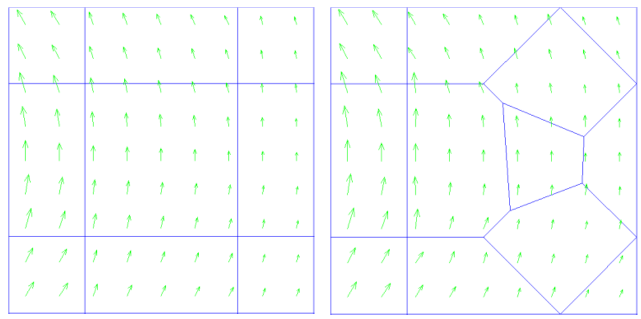

Fig. 4. Magnetic field interpolated on two different meshes through vector NEM shape functions. The source of this field is a punctual electric current flow placed at the point (-1.0, 0.0).

V. Conclusion

In this work a vector interpolation based on NEM was presented. The scheme was shown as being capable of interpolating different field configurations on meshes formed by different kinds of polygons. In the extended version of this paper the implementation, requirements and limitations of this approach will be explored. The effect of deformed polygons will be analyzed in terms of conditioning and accuracy [8]. Finally the application of the approach to the simulation of a real problem will be proposed.

VI. References

[1] J. Jin, The Finite Element Method in Electromagnetics. John Wiley & Sons, 2002.

[2] WEBB, J. P. Edge elements and what they can do for you. IEEE Trans. on Magnetics, v. 29, n. 2, p. 1460-1465, mar. 1993.

[3] N. Sukumar, B. Moran, T. Belytschko, “The Natural Element Method in Solid Mechanics”, International Journal for Numerical Methods in Engineering, vol. 43, pp. 839-887, 1998.

[4] E. Cueto, N. Sukumar, B. Calvo, M. A. Martínez, J. Cegoñino, M. Doblaré, “Overview and recent advances in natural neighbour Galerkin methods”, Archives of Computational Methods in Engineering, vol. 10, issue 4, pp. 307-384, 2003.

[5] R. Sibson, "A vector identity for the Dirichlet tessellation", Mathematical Proceedings of the Cambridge Philosophical Society, vol. 87, issue 1, pp. 151-155, 1980.

[6] P. Dular, R. Specogna, F. Trevisan, “Constitutive Matrices Using Hexahedra in a Discrete Approach for Eddy Currents”, IEEE Transaction on Magnetics, vol. 44, issue 6, pp. 694-697, 2008.

[7] J. Yvonnet, *Mise en œuvre de la méthode des éléments naturels contrainte en 3D : application au cisaillage adiabatique*, PhD dissertation, France : ENSAM Paris, 2008.

[8] A. Gillette, A, Rand, C. Bajaj, “Construction of Scalar and Vector Finite Element Families on Polygonal and Polyhedral Meshes”, arXiv:1405.6978v1, 2014.

[9] W. Chen, Y. Wang, “Minimal degree H(curl) and H(div) conforming finite elements on polytopal meshes”, arXiv:1502.01553v1, 2015.